Current behavior of a quantum Hamiltonian ratchet in resonance

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We investigate the ratchet current that appears in a kicked Hamiltonian system when the period of the kicks corresponds to the regime of quantum resonance. In the classical analogue, a spatial-temporal symmetry should be broken to obtain a net directed current. It was recently discovered that in quantum resonance the temporal symmetry can be kept, and we prove that breaking the spatial symmetry is a necessary condition to find this effect. Moreover, we show numerically and analytically how the direction of the motion is dramatically influenced by the strength of the kicking potential and the value of the period. By increasing the strength of the interaction this direction changes periodically, providing us with a non-expected source of current reversals in this quantum model. These reversals depend on the kicking period also, though this behavior is theoretically more difficult to analyze. Finally, we generalize the discussion to the case of a non-uniform initial condition.

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I. INTRODUCTION

The ratchet effect, a directed transport without any external net force, has attracted an increasing interest since early studies by Feynman et al.[1]. This phenomenon has a wide range of possible applications in rectifiers, pumps, particle separation devices, molecular switches, and transistors (see review articles [2, 3] and the references therein). It is also of great interest in biology, since the working principles of molecular motors can be conveniently explained in terms of ratchet mechanisms [4]. Moreover, it is possible to demonstrate quantum ratchet effects [5, 6, 7] by using cold atoms. As a result, many different scenarios have been considered [8, 9, 10, 11, 12, 13].

At the classical level, directed transport in periodic systems can be associated to a broken spatial-temporal symmetry [14]. We can obtain a net current by means of a periodically kicked system, for instance. In this situation one can break the spatial symmetry by using an asymmetric potential and the temporal symmetry by introducing dissipation [15] or an asymmetric kicking sequence [10]. In these cases the quantum versions present the same symmetry features as the classical counterparts, showing the corresponding current.

Current reversals is one of the interesting ratchet features that has attracted considerable interest [16, 17, 18, 19, 20]. Besides the ones due to symmetries in the potential, there are essentially two types of current reversals. One corresponds to dissipative chaotic systems and it is originated by bifurcations from chaotic to periodic regimes, that is, by the transition from strange attractors to simple ones [17]. The other [20] is explained by the fact that below certain temperatures, quantum tunneling can cause a change in the direction of transport.

The system that we consider in this paper shows directed transport associated to spatial asymmetry plus quantum resonance effects rather than to an explicit spatio-temporal symmetry breaking. This kind of systems was introduced in [21] where the authors have found a new mechanism for directed motion in quantum Hamiltonian systems. In this kind of systems momentum grows indefinitely, i.e. it does not stabilize around an asymptotic value (we could see this as a "rectification of force"). In their work they have shown that even if the system is time symmetric there can be transport in quantum resonance. Quantum resonance (QR) is a pure quantum phenomenon without a classical counterpart. In the well-known kicked rotor (KR) system,

$$H = -\frac{1}{2}\frac{\partial^2}{\partial\theta^2} + k\cos(\theta)\delta_T \qquad (1)$$

where $\delta_T = \sum_n \delta(t - nT)$, T is the period of the kick, k the strength of the kick, and $\hbar$, the Planck constant, and the moment of inertia of the rotator has been taken equal to 1). Given a value of the kick strength k, special resonant regimes of motion appear for periods with values $T = 4\pi\frac{r}{q}$, where the integers r and q are mutually prime. Under these conditions the system regularly accumulates energy which grows quadratically with the time and with k [22], that is $\langle p^2\rangle \propto k^2t^2$, where p stands for the momentum operator (for simplicity we refer to $\langle p\rangle$ as the momentum henceforth).

In the quantum KR at resonance (and for a symmetric or antisymmetric initial condition) there is no growth of the momentum, i.e., it is always equal to its initial value (though it does grow like $\langle p\rangle \propto kt$ if the initial condition is generic). In order to find a net current for any initial condition, a different potential has to be used. We prove that breaking any spatial symmetry is a necessary condition. The potential that we study corresponds to the

double well-kicked rotor (dw-KR)[10]. This potential has been experimentally realized in optical lattices [23].

In this model it is possible, for high resonances ($q > 2$), to have directed transport even for symmetric initial conditions. In addition, this modified KR at resonance shows a new kind of current reversals that (although being of quantum origin) are not due to tunneling (like in [20]). In fact, in our model the momentum will evolve as $\langle p\rangle \propto g(k)t$ where $g(k)$ is a non monotonic function of k. This is quite different from that case in the usual KR model. Moreover, we are able to give analytical predictions for the current reversals following a perturbative approach.

The paper is organized as follows. The main part of our paper, Section II, is devoted to the numerical and analytical study of the behavior of the current, including current reversals due to the variation of the kick strength and period. We study this phenomenon analytically for small values of the asymmetry. We also generalize this discussion to non-uniform initial conditions. In Section III we show that breaking the spatial symmetry is a necessary condition to find directed current at quantum resonance. In section IV we present our conclusions.

II. DIRECTED CURRENT BEHAVIOR

The Hamiltonian of the system is given by:

$$H = -\frac{1}{2}\frac{\partial^2}{\partial\theta^2} + k(\cos(\theta) + a\cos(2\theta+\alpha))\delta_T. \qquad (2)$$

where $\delta_T = \sum_n \delta(t - nT)$, T is the period and k is the strength of the kick. This Hamiltonian can be used to study a gas of cold atoms in an optical lattice. The difference between the dw-KR and the KR is due to the parameters a, that is the relative strength of the second harmonic and α, which is a parameter that breaks the symmetry of the kicking potential ($V(\theta) \neq V(2\pi - \theta)$ where $V(\theta) = k(\cos(\theta) + a\cos(2\theta + \alpha))$). The QR condition is given by $T = 4\pi r/q$, as in the KR case. Being $\hbar = 1$, an effective Planck's constant can be defined as $\hbar_{eff} = 8\omega_R T = T$. It is important to underline that there is no dissipation in our model. We study the problem on a torus and the initial condition is $\phi_0 = 1/\sqrt{2\pi}$, unless otherwise stated. This can be implemented on an optical lattice with a wavefunction almost uniformly spread on many lattice sites. The choice of this initial condition is due to the fact that it is symmetric. This leads to the non typical behavior of this system compared with the usual KR where for this scenario the momentum would stay always at zero. Incidentally, it makes the analytical computations much easier.

Firstly, we would like to make reference to the inset of Fig.1, where we can see that for a given k, the momentum $\langle p\rangle$ grows linearly with time, that is with the number of kicks N ($\langle p\rangle \propto t \propto N$). For this numerical evaluation we chose $T = 4\pi\ 1/3$, $k = 5$, $a = 0.01$ and $\alpha = \pi/4$. Since $\langle p\rangle$ grows linearly with the number of kicks N we focus our attention on a quantity defined as $\langle f\rangle = \langle p\rangle/N$, which we call average effective force. In Fig.1 we show the average effective force versus the kick strength. We can see that there is a general growth of $\langle f\rangle$ with k, but more interestingly, we also see that $\langle f\rangle$ oscillates radically, going from positive to negative values, i.e., there is current inversion. To obtain this set of values we have used $a = 2$, representative of the behavior of $\langle f\rangle$ for large values of a, and $\alpha = \pi/4$. In order to see if this behavior can be detected in an experiment with cold atoms, we have evaluated $\langle p\rangle/\sqrt{\langle p^2\rangle - \langle p\rangle^2}$. For this set of parameters, we have found that it saturates to the value 0.18 at about 15 kicks ($N = 15$), guaranteeing the feasibility of a future realization.

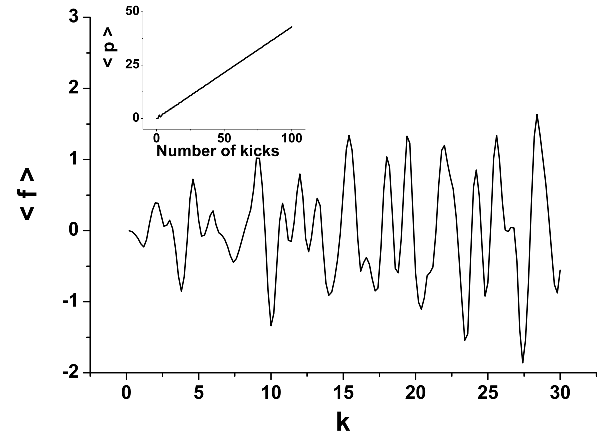

FIG. 1: The average effective force $< f >$ as a function of the kick strength k, for $r/q = 1/3$, $a = 2$ and $\alpha = \pi/4$. In the inset we show the momentum $\langle p\rangle$ versus the number of kicks for $k = 5$, $r/q = 1/3$, $a = 0.01$ and $\alpha = \pi/4$.

In the following we will show the origin of the current inversions that occur for different values of the kick intensity or the period. To do this we do a perturbative study for small values of the parameter a, which is the amplitude of the second harmonic. In Ref. [22] it has been shown that the one kick evolution of an initial condition $\phi_0(\theta)$ is given by:

$$\phi_1(\theta) = \sum_{n=0}^{q-1} S_{0n}\phi_0(\theta + \frac{2\pi n}{q}) \qquad (3)$$

where S_{0n} are given by $\beta_0(\theta)\gamma_n$ with $\beta_n = \exp(-iV(\theta + \frac{2\pi n}{q}))$ and $\gamma_n = \sum_{m=0}^{q-1}\exp(-i\frac{2\pi r m^2}{q} - i\frac{2\pi m n}{q})$. Starting with a uniform initial condition we have:

$$\phi_1(\theta) = \beta_0 \sum_{l_0=0}^{q-1}\gamma_{l_0}\phi_0(\theta + \tfrac{2\pi l_0}{q}) = \tfrac{\beta_0}{\sqrt{2\pi}}$$
$$\sqrt{2\pi}\phi_2(\theta) = \beta_0\sum_{l_1=0}^{q-1}\gamma_{l_1}\beta_{l_1} \qquad (4)$$

So, after a generic kick N the wavefunction ϕ_{N+1} is given

by:

$$
\begin{array}{c}
\sqrt{2\pi}\phi_{N+1}(\theta) = \\
= \beta_0 \sum_{l_1,\ldots l_N=0}^{q-1} \gamma_{l_1}\beta_{(l_1+l_2+\ldots l_N)} \times \gamma_{l_2}\beta_{(l_2+\ldots l_N)} \ldots \times \gamma_{l_N}\beta_{l_N} \\
= \beta_0 \sum_A \prod_{m,n=1}^{N} \gamma_{l_m}\beta_{l_n}
\end{array}
\tag{5}
$$

where A stands for the combinations given in the previous line of this equation.

The momentum at time $N+1$ is given by $\langle p_{N+1}\rangle = \int_0^{2\pi} \phi^*_{N+1}(-i\frac{d}{d\theta}\phi_{N+1})d\theta$ ($\hbar = 1$) giving:

$$
\begin{array}{c}
\langle p_{N+1}\rangle = \frac{-i}{2\pi}\int_0^{2\pi} \left(\beta_0^* \sum_A \prod_{m,n=1}^{N} \gamma^*_{l_m}\beta^*_{l_n}\right) \\
\left(\frac{d}{d\theta}\left(\beta_0 \sum_A \prod_{m,n=1}^{N} \gamma_{l_m}\beta_{l_n}\right)\right)d\theta
\end{array}
\tag{6}
$$

which can be rewritten as:

$$
\begin{array}{c}
\langle p_{N+1}\rangle = \frac{-i}{2\pi}\int_0^{2\pi} \left(\beta_0^* \sum_A \prod_{m,n=1}^{N} \gamma^*_{l_m}\beta^*_{l_n}\right) \\
\left(\beta_0' \sum_A \prod_{m,n=1}^{N} \gamma_{l_m}\beta_{l_n} + \beta_0 \sum_A \left(\frac{d}{d\theta}\left(\prod_{m,n=1}^{N} \gamma_{l_m}\beta_{l_n}\right)\right)\right)d\theta
\end{array}
\tag{7}
$$

As we have shown in the inset of Fig.1 the momentum grows linearly with N. The only part of the previous expression that satisfies this condition is:

$$
\begin{array}{c}
\langle p_{N+1}\rangle \approx \frac{-i}{2\pi}\int_0^{2\pi} \left(\beta_0^* \sum_A \prod_{m,n=1}^{N} \gamma^*_{l_m}\beta^*_{l_n}\right) \\
\left(\beta_0 \sum_A \left(\frac{d}{d\theta}\left(\prod_{m,n=1}^{N} \gamma_{l_m}\beta_{l_n}\right)\right)\right)d\theta
\end{array}
\tag{8}
$$

because we have the derivative of a product of N terms. Hence, we can approximate Eq.(6) by:

$$
\begin{array}{c}
\langle p_{N+1}\rangle \approx N\frac{-i}{2\pi}\Big(\int_0^{2\pi} \left(\sum_{m\neq n} \gamma^*_m\beta^*_m\gamma_n\beta'_n\right)d\theta + \\
+ \int_0^{2\pi} \left(\sum_{l,m,n,s} \gamma^*_l\gamma^*_m\gamma_n\gamma_s\beta^*_{l+m}\beta^*_m\left(\beta_{n+s}\beta_s\right)'\right)d\theta + \ldots\Big)
\end{array}
\tag{9}
$$

Here we make a further approximation, we drop all the integrals with the exception of the first one. This is justified by the fact that their integrands are highly oscillating functions of θ compared to the first one, so their contribution to the final result is negligible.

In order to simplify the analytic treatment without losing the essential features of $\langle p\rangle$ that we want to describe, we focus on the usual KR perturbed by a second harmonic, i.e., we take $k \sim O(1)$ and $a \ll 1$. In this case, as long as $ka \ll 1$ we can approximate $\beta_m \simeq \tilde{\beta}_m = \exp(-ik\cos(\theta + 2\pi m/q))(1 + ika\cos(2\theta + 4\pi m/q + \alpha))$. Then, Eq.(9) becomes:

$$
\langle p_{N+1}\rangle \approx N\frac{-i}{2\pi}\int_0^{2\pi} \left(\sum_{m,n} \gamma^*_m\gamma_n V'(\theta + 2\pi n/q)\tilde{\beta}^*_m\tilde{\beta}_n\right)d\theta
\tag{10}
$$

These terms are all integrable and after a few computations we can write the momentum as $\langle p_{N+1}\rangle \approx \sum_{m,n} L_{m,n}$ where $L_{m,n}$ is given by:

$$
\begin{array}{c}
L_{m,n} = k\ a\ N\gamma^*_m\gamma_n \\
\Big(\sin\left(\frac{2\pi n}{q} - \omega_{m,n}\right)\times \\
\times\left[\cos\left(\frac{4\pi n}{q} + \alpha - 2\omega_{m,n}\right) - \cos\left(\frac{4\pi m}{q} + \alpha - 2\omega_{m,n}\right)\right]\times \\
\times\left(k\ J_1(\omega_{m,n}k) - 2\ J_2(\omega_{m,n}k)/\omega_{m,n}\right) + \\
-\cos\left(\frac{2\pi n}{q} - \omega_{m,n}\right)\times \\
\times\left[\sin\left(\frac{4\pi n}{q} + \alpha - 2\omega_{m,n}\right) - \sin\left(\frac{4\pi m}{q} + \alpha - 2\omega_{m,n}\right)\right]\times \\
\times 2\ J_2(\omega_{m,n}k)/\omega_{m,n} + \\
-2\sin\left(\frac{4\pi n}{q} + \alpha - 2\omega_{m,n}\right)J_2(\Omega_{m,n}k)\Big)
\end{array}
\tag{11}
$$

where $\Omega_{m,n} = \sqrt{\mu^2_{m,n} + \nu^2_{m,n}}$ and $\tan\omega_{m,n} = \left(\frac{\nu_{m,n}}{\mu_{m,n}}\right)$. Here we have used $\mu_{m,n} = [\cos(2\pi n/q) - \cos(2\pi m/q)]$ and $\nu_{m,n} = [\sin(2\pi n/q) - \sin(2\pi m/q)]$. The momentum $\langle p\rangle$ depends on the period parameter r through the coefficients γ_m. We now see that it is not surprising to find current inversion. In fact, we expect a different sign of the average effective force for different values of k, because the Bessel functions oscillate.

From the properties of the Bessel functions we can understand the behavior for small and large k. For the case of $k \gg 1$, we must guarantee that $ka \ll 1$. In general we have that for $x \ll 1$, $J_\alpha(x) \simeq \frac{1}{\Gamma(1+\alpha)}\left(\frac{x}{2}\right)^\alpha$ (Γ is the function which generalizes the factorial to non-integer numbers) and for $x \gg 1$ we have that $J_\alpha(x) \simeq \sqrt{\left(\frac{2}{\pi x}\right)}\cos(x - \alpha/4 - \pi/2)$. For higher values of the period parameter q ($T = 4\pi\ r/q$), the situation is more complex because we have many different $\Omega_{m,n}$ and so there is a superposition of many terms like $\cos(\Omega_{m,n}k - \alpha\pi/4 - \pi/2)$. This is much more difficult to study analytically.

In general, for large values of k, we can approximate the behavior by

$$
\begin{array}{c}
\langle p_{N+1}\rangle \approx \sum_{m,n}\left[A\left(\frac{k^3}{\Omega_{m,n}}\right)^{1/2}\cos(\Omega_{m,n}k - \frac{3\pi}{4}) + \right. \\
\left. B\left(\frac{k}{\Omega_{m,n}}\right)^{1/2}\cos(\Omega_{m,n}k - \frac{5\pi}{4})\right],
\end{array}
\tag{12}
$$

where A and B are two constants that can be obtained from Eq.(11). We can think of this expression as divided into two parts. The first part dominates for large values of k when it is far enough from its zeros. Even if we consider only the first part, the sum of many cosine functions with different periods gives a very fluctuating behavior.

To confirm the above analysis, we compute the particular cases $r/q = 1/3$ and $r/q = 1/5$ and compare them with the numerical solution obtained from the evolution of the wavefunction. In the case of $r/q = 1/3$, $\Omega_{m,n}$ takes only two values, either 0 or $\sqrt{3}$ and so we only have terms like $J_\eta(\sqrt{3}k)$ where $\eta = 1, 2$.

After some analytical computations we can see that the average effective force is given by:

$$
\begin{array}{c}
\langle f\rangle = k\ a\ \sin(\alpha) \\
\left[\left(\frac{1}{\sqrt{3}} - 1\right)k\ J_1\left(\sqrt{3}k\right) + \frac{2}{3}\left(1 + \sqrt{3}\right)\ J_2\left(\sqrt{3}k\right)\right]
\end{array}
\tag{13}
$$

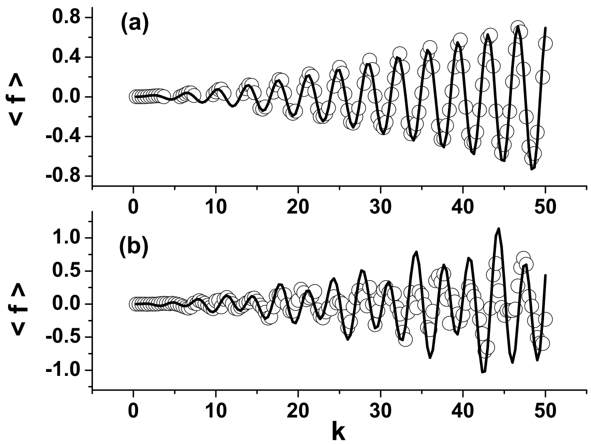

FIG. 2: Average effective force $\langle f\rangle$ versus k for $a = 0.01$ and $\alpha = \pi/3$. Results from the numerical evolution of the wavefunction (circles) and the analytical approximation (solid line) are compared. In (a) $T = 4\pi\ 1/3$ while in (b) $T = 4\pi\ 1/5$. We can see the oscillations showing current reversals.

The case for $r/q = 1/3$ is shown in Fig.2(a) where the numerical (circles) and analytical (solid line) results are compared and show good agreement. A different period implies different values of Ω_{ij} and so different values of the period of oscillation for the same value of k. We should expect very different behavior for different r/q. This is clarified in Fig.2(b) where we plot the numerical (circles) and analytical (solid line) results showing the average effective force versus k for $r/q = 1/5$. In this case, differences are due to the cut-off of oscillating terms.

It is interesting to notice one important difference in this model compared to the usual quantum KR model. In our model the average effective force does not grow linearly with the intensity of the kick. Instead, it oscillates periodically. Even the peaks do not show this linear behavior with respect to k. There are two components contributing to the average effective force. One of them causes the peak to grow with a power 1/2 ($\langle f\rangle \propto k^{1/2}$, i.e. $\langle p\rangle \propto k^{1/2}t$) and the other makes it to grow with a power 3/2 ($\langle f\rangle \propto k^{3/2}$).

Reversals due to changes in the value of the kick strength k have been found previously [24], but in a completely different context. Here we find reversals due to changes in k and also in the value of the period, i.e. in the effective Planck constant. Since in our case the nature of the current is different from the one in [24], these inversions are also of a different and novel character [13]. An insightful model of QRs in the quantum KR can be found in [25], and we think that a theoretical explanation for this phenomenon could be investigated following those lines.

In the generic case, the initial condition will not be symmetric, a situation which is most likely to happen in real experimental conditions. As already discussed, for an asymmetric initial condition and a symmetric kick, quantum resonance induces directed motion and the momentum will increase linearly with the time and with the strength of the kick. On the other hand, with a non symmetric kick in quantum resonance, we will see the oscillating behavior that we have found previously, superimposed in this case to a linear growth due to the asymmetry of the initial condition. This is confirmed by the numerical results shown in Fig. 3. We have considered $T = 4\pi\ 1/3$, $a = 0.01$, $\alpha = \pi/3$ and the initial condition $\phi_0(\theta) = \eta\cos(\cos(\theta) + \sin(2\theta))$, where η is the normalization constant.

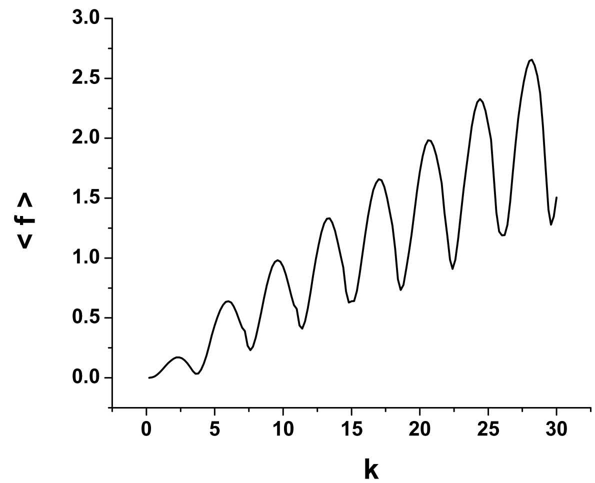

FIG. 3: Average effective force $\langle f\rangle$ versus k for $T = 4\pi\ 1/3$, $a = 0.01$ and $\alpha = \pi/3$. The initial condition is $\phi_0(\theta) = \eta\cos(\cos(\theta) + \sin(2\theta))$, where η is the normalization constant.

III. NECESSARY CONDITION FOR DIRECTED TRANSPORT IN QUANTUM RESONANCE

In this section, we shall show that in the absence of a net force, either an asymmetry in the initial condition or in the kicking potential is necessary to have directed transport. It should be stressed that these are necessary but not sufficient conditions, because it is possible to have zero transport also when at least one of these two asymmetries is present. Representing a gas of cold atoms with the wavefunction ϕ, the momentum is given by $\langle p\rangle = -i\hbar\int_0^{2\pi}\phi^*\left(\frac{d}{d\theta}\phi\right)d\theta$. Let's take again a generic kicked system with a Hamiltonian given by:

$$H = -\frac{1}{2}\frac{\partial^2}{\partial\theta^2} + V(\theta)\delta_T \qquad (14)$$

where $\delta_T = \sum_n \delta(t - nT)$, T is the period of the kick and V is the shape of the kicking potential. We assume that the system is at quantum resonance ($T = 4\pi\frac{r}{q}$ with

r and q relative prime numbers). Given the wavefunction after N kicks (ϕ_N), we can obtain the wavefunction after $N+1$ kicks $\phi_N + 1$ by [22]:

$$\phi_{N+1}(\theta) = \beta_0 \sum_n \gamma_n \phi_N \left(\theta + 2\pi \frac{n}{q}\right) \tag{15}$$

where $\beta_n = \exp(-iV(\theta + \frac{2\pi n}{q}))$ and $\gamma_n = \sum_{m=0}^{q-1} \exp(-i\frac{2\pi r m^2}{q} - i\frac{2\pi m n}{q})$. Using the definition of γ one can verify that

$$\gamma_n = \gamma_{n+q} = \gamma_{q-n}. \tag{16}$$

Let's study the case with a symmetric initial condition ($\phi_N(2\pi-\theta) = \phi_N(\theta)$) and a symmetric kicking potential ($V(2\pi-\theta) = V(\theta)$). In this case we can see that $\beta_n(2\pi-\theta) = \beta_{q-n}(\theta)$ and then

$$\begin{aligned} \phi_{N+1}(2\pi-\theta) &= \beta_0(2\pi-\theta) \textstyle\sum_n \gamma_n \phi_N \left(2\pi - \theta + 2\pi\frac{n}{q}\right) \\ \phi_{N+1}(2\pi-\theta) &= \beta_0(\theta) \textstyle\sum_n \gamma_{q-n} \phi_N \left(\theta + 2\pi \frac{q-n}{q}\right). \end{aligned} \tag{17}$$

This shows that given a symmetric initial condition and a symmetric kicking potential, the wavefunction will always be symmetric and therefore the momentum will always be zero. This result is readily extended to anti-symmetric initial conditions.

We now show that if the initial condition has a shape like $\phi_N(\theta) = e^{iL\theta} f_N(\theta)$ where $f_N(\theta) = f_N(2\pi - \theta)$ is even, the kick will not change the value of the momentum which is $\langle p \rangle = L$. For example, an eigenstate of the momentum $\phi_0 = \frac{e^{iL\theta}}{\sqrt{2\pi}}$ will not change its direction or speed. Using Eq.(15) it is easy to show that:

$$\begin{aligned} \phi_{N+1}(\theta) &= \beta_0(\theta) \textstyle\sum_n \gamma_n e^{iL(\theta + 2\pi\frac{n}{q})} f_N\left(\theta + 2\pi\frac{n}{q}\right) = \\ &= \beta_0(\theta) e^{iL\theta} \textstyle\sum_n \gamma_n e^{iL(2\pi\frac{n}{q})} f_N\left(\theta + 2\pi\frac{n}{q}\right) \end{aligned} \tag{18}$$

where we have factorized out the $e^{iL\theta}$ which contributes to the momentum and now the factors γ_n are multiplied by a phase $e^{iL(2\pi\frac{n}{q})}$. The symmetry of $\phi_{N+1}(\theta)$ is given by:

$$\begin{aligned} &\phi_{N+1}(2\pi-\theta) = \\ &= \beta_0(2\pi-\theta) \textstyle\sum_n \gamma_n e^{iL(2\pi-\theta+2\pi\frac{n}{q})} f_N\left(2\pi - \theta + 2\pi\frac{n}{q}\right) = \\ &= \beta_0(\theta) e^{iL(2\pi-\theta)} \textstyle\sum_n \gamma_{q-n} e^{iL(2\pi\frac{n}{q})} f_N\left(\theta + 2\pi\frac{q-n}{q}\right) = \\ &= \beta_0(\theta) e^{iL(2\pi-\theta)} \textstyle\sum_n \gamma_{q-n} e^{iL(2\pi\frac{q+n}{q})} f_N\left(\theta + 2\pi\frac{q-n}{q}\right) \end{aligned} \tag{19}$$

Using the fact that $\gamma_n e^{iL(2\pi\frac{n}{q})} = e^{-iL^2(2\pi\frac{p}{q})} \gamma_{n+2L} = e^{-iL^2(2\pi\frac{p}{q})} \gamma_{q-n+2L} = e^{-iL^2(2\pi\frac{p}{q})} \gamma_{q-n-2L}$ it is possible to show that:

$$\gamma_n e^{iL(2\pi\frac{n}{q})} = \gamma_{q-n} e^{iL(2\pi\frac{q-n}{q})} \tag{20}$$

From Eq.(20) it is evident that after any kick $\phi_N(\theta)$ can be decomposed as $\phi_N(\theta) = e^{iL\theta} f_N(\theta)$ and this proves that the momentum will not change.

It is well known that with an asymmetric initial condition we can have transport. In fact, for the simple case $T = 4\pi$, we have that $\phi_N = e^{-iNV(\theta)}\phi_0$. If ϕ_0 is not symmetric, even if we have zero net force the momentum grows linearly with the number of kicks. In [21], it has been found that with a non-symmetric kick, transport can be induced by high q resonances. This means that it is necessary to at least break the spatial symmetry to have transport. To do this we can either start with a non-symmetric initial condition, use a non-symmetric kick or a combination of the above situations.

IV. CONCLUSIONS

We have analyzed in detail the phenomenon of directed transport for a Hamiltonian system in quantum resonance. We have found that directed transport can be in different directions depending on the intensity of the kick or on its period. This phenomenon was unexpected and presumably due to how the intensity of the kick and the value of the period affect the gradient of the phase of the wavefunction. Very interestingly, the direction of the motion changes periodically with the intensity of the kick. Also remarkable is the fact that the momentum at the peaks does not increase linearly with the intensity of the kick, but it follows one term that goes as a square root and another that goes as a power law with exponent 3/2. We have also shown that for an asymmetric initial condition the periodic effect due to quantum resonance is superimposed to the expected drift due to the asymmetry of the initial condition. Finally, we have found that even though an explicit time symmetry breaking is not needed, breaking the spatial symmetry is a necessary condition to have directed transport. By numerical simulations we have also found out that this effect is very sensitive on the initial condition. To see clearly these effects experimentally a non-interacting BEC could be used because the initial wavefunction is less-spread in the momentum space than for a gas of cold atoms.

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